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Efficient Dual-Sphere Microphone-Array Design Based on Generalized Sampling Theory

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Summary

The generalized sampling expansion, introduced by Papoulis, facilitates the reconstruction of a band-limited signal that has been sampled at a rate slower than the Nyquist rate - provided that additional information about the signal is available in the form of the sampled outputs of known linear systems with the original signal at their input. In this work, the generalized sampling expansion, originally developed for time-domain signals, is formulated for functions over the sphere, using the spherical harmonics transform. The paper presents the theory of the new expansion that has been developed for the equal-angle sampling scheme. In the second part of the paper, the theory is applied to the design of an efficient dual-sphere microphone-array. A known problem, when sampling sound fields using an open-sphere microphone-array, is the ill-conditioning at specific frequencies due to the nulls of the spherical Bessel function. To overcome this problem, a dual-sphere design, which requires twice as many microphones compared to a single-sphere design, has previously been proposed. Applying the generalized sampling theory developed here, it is shown that a dual-sphere design with half the number of samples at each sphere can replace a single sphere, but only if the two spheres are rotated relative to each other in a specific manner. Reconstruction of the sound pressure on the sphere is then possible without increasing the total number of microphones, while at the same time countering the effect of the nulls.

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1. Introduction

Spherical microphone-arrays are useful for many applications, such as directivity measurement of musical instruments [1], calculating the directional impulse response of rooms in order to study the acoustic performance of auditoria [2] and spatial recording for 3D sound reproduction purposes [3]. For obvious practical reasons, when measuring a sound field on a sphere, the field is sampled using a finite number of microphones. Driscoll and Healy [4] presented the sampling theory on a sphere and showed that, similar to the Shannon sampling theory in a Cartesian coordinate system, a function on a sphere can be perfectly reconstructed if the maximum spectral order is lower than a specified limit, which is determined by the number and the location of the samples.

A common method used to describe sound fields is to express the amplitude of a sound field using the spherical harmonics representation. However, for the case where the radius and the wave-number give rise to nodes in the radial function, e.g. open-sphere configuration [2] the directivity information cannot be computed by sampling. One way to overcome this problem is to use a dual-sphere configuration [5], where two sphere are employed in order to avoid such nodes. This method is based on selecting one of the two spheres such that the amplitude of the radial function is maximized for each wave number.

The dual-sphere method increases the robustness of an array as its capability of sampling and reconstruction of functions with wave-numbers that give rise to nodes when using the single sphere method. However, the maximum spectral order that can be sampled without error does not increase, despite the additional microphones, suggesting that a more efficient method in terms of spectral order may exist.

This paper proposes an efficient dual-sphere method that requires the same number of microphone as a single-sphere method. Firstly, in order to support the reasoning of the new method, the theory of generalized sampling developed in this paper for functions on the sphere is introduced. The well known sampling theorem introduced by Shannon states that in order to reconstruct a band-limited function it should be sampled at a rate higher than the Nyquist rate [6]. Papoulis [7] presented the generalized sampling expansion, showing that a band-limited function is reconstructable from V linear, time invariant (LTI) systems that are each sampled at $1/V$ the Nyquist rate. The theory shows that in order to increase the maximum spectral order of a function that can be sampled without error, one sphere should be rotated relative to the other.

The proposed methods works well for equal-angle and Gaussian sampling schemes, which are widely used schemes in room acoustics applications [8].

2. Sampling and aliasing in Spherical Coordinate Systems

In this section, two topics that are important to the work presented in this paper are reviewed; i) linear systems and ii) sampling and aliasing in spherical coordinates systems. Then, generalized sampling for functions on the sphere is introduced.

2.1. Linear systems in Spherical Coordinates

The standard spherical coordinate system (r, θ, ϕ) defined in [9], and the Fourier transform on a sphere as defined in [4] read:

$$p(\theta, \phi) = \sum_{q=0}^{\infty} p_q Y_q(\theta, \phi) \tag{1}$$

$$p_q = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} p(\theta, \phi) Y_q(\theta, \phi)^* \sin(\theta) \mathrm{d}\phi \mathrm{d}\theta \tag{2}$$

where $Y_q(\theta, \phi)$ are the spherical harmonics defined in [4] and $q = n^2 + n + m$, where n is the order and m is the degree of the spherical harmonics. The use of a single index q rather than the more conventional indexes n and m is chosen for notation simplicity. The summation in (1) denotes a double summation for $n = 0, ..., \infty$ and for $-n <= m <= n$.

If there exists an integer Q such that $\forall q \geq Q$, $p_q = 0$, the function is called order limited. The infinite sum in equation (1) for order-limited functions reduces to:

$$p(\theta, \phi) = \sum_{q=0}^{Q-1} p_q Y_q(\theta, \phi) \tag{3}$$

We will refer to a linear system as an operator $\mathcal{H}$ acting on order-limited signals on a sphere of order Q, $\mathcal{H} : L^2(S_Q^2) \rightarrow L^2(S_Q^2)$, where the following relation between the input p and the output g is satisfied

$$g = \mathcal{H}p \rightarrow g_q = \sum_{q'=0}^{Q-1} h_{qq'} p_{q'} \tag{4}$$

Equation (4) can be represented in a matrix form

$$\mathbf{g} = \mathbf{H}\mathbf{p} \tag{5}$$

where $\mathbf{p}$ and $\mathbf{g}$ are $Q \times 1$ vectors of coefficients p_q and g_q respectively and $\mathbf{H}$ is a $Q \times Q$ matrix of coefficients $h_{qq'}$.

There are several types of systems which can be represented in the above form. Driscoll and Healy [4] proved that spherical convolution of a function in S^2 acting on another functions in S^2 can be represented in the spherical harmonics domain as a system in which matrix $\mathbf{H}$ in (5) is diagonal. Healy et al. [10] showed that the rotation of a function can be represented as a system in which $\mathbf{H}$ is block diagonal. In [11] the translation operation is developed, deriving the formula of a translation matrix $\mathbf{H}$ having the same from as in (5).

2.2. Sampling and Aliasing in spherical coordinates

The sampling of functions on a the sphere is investigated by Driscoll and Healy [4]. A function on a sphere is reconstructed from a set of sampling point $p(\theta_l, \phi_l), l = 0...L-1$ [8]

$$\hat{p}_q = \sum_{l=0}^{L-1} \alpha_l p(\theta_l, \phi_l) Y_q^*(\theta_l, \phi_l) \tag{6}$$

where α_l are the quadrature weights determined by the sampling scheme and are chosen such that for a function of order Q smaller than the limit imposed by the sampling schemes, the reconstruction is exact, i.e. $\hat{p}_q = p_q \; \forall q < Q$. Three common sampling schemes are the equiangular, Gaussian and nearly uniform sampling [5]. An equiangular sampling scheme of order N is defined by the set of point $p(\theta_k, \phi_j)$ where

$$\theta_k = \pi k / 2(N+1), k = 0, ..., 2N+1 \tag{7a}$$

$$\phi_j = 2\pi j / 2(N+1), j = 0, ..., 2N+1 \tag{7b}$$

Thus for a sampling configuration of order N the number of sampling points is $4(N+1)^2$. In order to reconstruct an order-limited function on a sphere, all coefficients $p_q, q = 0, ..., Q-1$ are required to be computed. Substituting (3) in (6) yields

$$\hat{p}_q = \sum_{q'=0}^{Q-1} p_{q'} \sum_{l=0}^{L-1} \alpha_l Y_{q'}(\theta_l, \phi_l) Y_q^*(\theta_l, \phi_l) \tag{8}$$

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If the summation on the right-hand-side of (8) equals to $\delta_{qq'}$ then $\hat{p}_q = p_q$ and thus the function can be fully reconstructed. However, when the function order Q is larger than the limit imposed by the sampling scheme then $\hat{p}_q$ is generally different from p_q due to aliasing. The relation between the original coefficients p_q and the reconstructed coefficients can be written in a matrix form:

$$\hat{\mathbf{p}} = \mathbf{A}\mathbf{p} \tag{9}$$

where $\hat{\mathbf{p}}$ is the reconstructed coefficient vectors $[\hat{p}_0, \dots, \hat{p}_{Q-1}]^T$, $\mathbf{p}$ is the original coefficients vector $[p_0, \dots, p_{Q-1}]^T$ and $\mathbf{A}$ is the matrix determining the linear relation between the two vectors, which using (8) equals to

$$\mathbf{A}_{qq'} = \sum_{l=0}^{L-1} \alpha_l Y_{q'}(\theta_l, \phi_l) Y_q^*(\theta_l, \phi_l)$$

and is referred to as the aliasing matrix. When the reconstruction is exact, $\mathbf{A}$ is the identity matrix of size $Q \times Q$, and $\hat{\mathbf{p}} = \mathbf{p}$. However, when the sampled function is of order higher than the limit imposed by the sampling scheme, the reconstructed coefficients differ from the original ones. It seems that the effect of aliasing can be corrected by multiplying $\hat{\mathbf{p}}$ by the inverse of $\mathbf{A}$. Unfortunately, as shown in the appendix, when the function is of an order higher than the order imposed by the sampling scheme the matrix is singular due to aliasing.

2.3. Generalized Sampling Expansion in Spherical Coordinates

The generalized sampling expansion (GSE) developed by Papoulis [7], facilitates the reconstruction of a band-limited function $f(t)$, by sampling the outputs of V LTI systems with input $f(t)$ at a rate that is $1/V$ the Nyquist rate. Two common examples provided by Papoulis are i) a scheme where the function and the first $V-1$ derivatives are given at every sampling point and ii) a series of delayed systems. For a detailed study of the GSE in Cartesian coordinate system, the reader is referred to [7, 12].

We now use sections 2.1 and 2.2 in order to derive the GSE in spherical coordinates. A diagram of the suggested reconstruction scheme is presented in figure 1. An order-limited function $p(\theta, \phi)$ with spherical coefficients p_q is the input to a series of V linear systems with spherical coefficients $h^v_{qq'}$. These functions are then sampled, and the original function f is finally reconstructed using a series of postfilters B^v, $v = 1, ..., V$. Let g^v_q be the output of the v-th linear system in the spherical harmonics domain as given by equation (4). The sampling process can now be written in the spherical harmonics domain, by combining (5) and (9), as

$$\hat{\mathbf{g}}^v = \mathbf{A}\mathbf{H}^{\mathbf{v}}\mathbf{p} \tag{10}$$

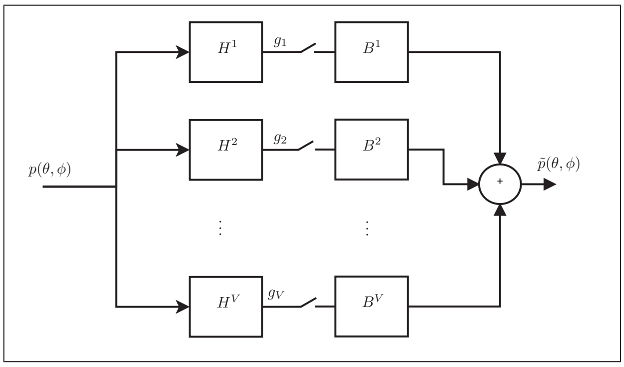

Figure 1. Generalized sampling and reconstruction diagram in spherical harmonics

Matrix $\mathbf{AH}^v$ is square, and thus if it has full rank the coefficients p_q can be computed by solving (10). However, due to aliasing, the rows of matrix $\mathbf{AH}^v$ are generally linearly dependent so that the function can not be reconstructed using a single system. A possible way to overcome the rank deficiency of matrix $\mathbf{AH}^v$ involves concatenation of equation (10) for all v:

$$\begin{bmatrix} \hat{\mathbf{g}}^\mathbf{1} \\ \hat{\mathbf{g}}^\mathbf{2} \\ \vdots \\ \hat{\mathbf{g}}^\mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{AH}^1 \\ \mathbf{AH}^2 \\ \vdots \\ \mathbf{AH}^V \end{bmatrix} \mathbf{p} = \mathbf{AHp} \tag{11}$$

If $\mathbf{AH}$ has full rank, then the postfilters can be calculated using the pseudo inverse operator:

$$\mathbf{B} = [\mathbf{AH}]^\dagger \tag{12}$$

which solves the overdetermined system in a least squares sense. Computation of the exact number of systems V required in order to solve (11) is beyond the scope of this paper and may be application-dependent. However, a lower bound is given by $VQ_s \geq Q$, where Q is the function order and Q_s is the maximum order which can be sampled without error with one system alone.

3. Efficient Dual-Sphere Design

3.1. The Dual-Sphere Array Configuration

The theory of sampling of functions on the sphere has been employed in the design of spherical microphone arrays for the analysis of measured sound fields in auditoria. One of the sampling schemes proposed is referred to as the dual-sphere method [2].

The so-called "dual-sphere method" is a sampling scheme in which microphones are placed at the surface of two spheres of different radii using the same sampling configuration, as illustrated in figure 2. The purpose of this scheme is to overcome the numerical ill-conditioning at wave numbers that correspond to nulls of the radial function [5]. Next we consider a

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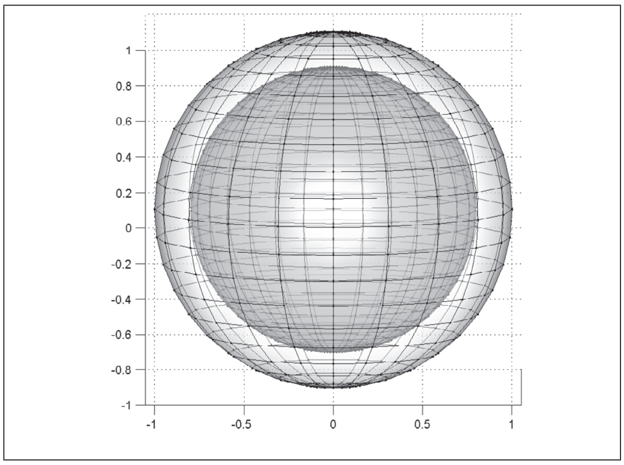

Figure 2. The dual sphere configuration.

sound field in three-dimensions. When all sources in the sound field are located outside an open sphere of radius r centered at the origin of a spherical coordinate system, than the solution for the Helmholtz equation is given by [13]

$$p(kr,\theta,\phi) = \sum_{q=0}^{\infty} c_q j_n(kr) Y_q(\theta,\phi) \tag{13}$$

where $j_n(kr)$ are the spherical Bessel functions and n is the spherical order corresponding to q. Using (2), the spherical coefficients c_q can be computed

$$c_q = \frac{p_q}{j_n(kr)} \tag{14}$$

Equation (14) shows that the coefficients c_q can be computed from p_q, which in turn can be computed from samples of p using (6). However, for certain kr such that $j_n(kr)$ is zero or relatively small, numerical inaccuracies may occur in the computation of c_q. One possible solution to this problem involves the use of samples covering the surface of two spheres, with different radii, such that for every wave-number in the operating range of thee system, the value of c_q is calculated using data measured at the sphere that exhibits the best condition number at the given wave-number. Balmages and Rafaely [5] showed that a ratio of about 1.2 between the radii of the two spheres is near optimal for a wide range of arrays in terms of the condition number.

Although the dual-sphere method overcomes the problem of reduced robustness due to the nulls of the radial function, it does require twice as many microphones (or sampling points) as compared to the single-sphere method. Next, we show how the GSE can be employed in order to increase the efficiency of the dual-sphere method by effectively combining the information obtained from the two spheres, while using half the number of samples.

3.2. Generalized Sampling Expansion for the Dual-Sphere Configuration

We now apply the GSE presented in section 2.3 for the dual sphere configuration, considering each sphere as a separate linear system. Let c_q be a set of coefficients of an order-limited function such that $c_q = 0 \; \forall q \geq Q$. (14) states that $p_q = j_n(kr)c_q$, hence, the spheres can be represented as a scaling linear system with $h_{qq'} = \delta_{qq'} j_n(kr)$, and the theory developed in section 2.3 can be used for this case.

However, for this case, matrix $[\mathbf{AH}]$ would not be of full rank when applying the GSE. This is due the following: if we assume that the matrix $\mathbf{AH}^1$ is not of full rank due to aliasing, then there are at least two columns which are linearly dependent. For example, consider an equiangular sampling configuration of $Q = 0$ and a function of order $Q = 4$, with a wave number k such that $kr_1 = 1.528$. Such a sampling scheme is designed to sample functions of order zero (i.e composed of Y_0) while the sampled function is of order $Q = 4$ (i.e Y_0 to Y_3, monopoles and dipoles) and dipoles, so that aliasing is expected. Matrix $\mathbf{AH}^1$ for this case is

$$\mathbf{AH}^1 = \begin{bmatrix} 0.65 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & -0.3 \\ 0 & 0 & 0.6 & 0 \\ 0 & -0.3 & 0 & 0.3 \end{bmatrix} \tag{15}$$

One can easily see that the second column is equal to the forth column times minus one due to aliasing, so that the matrix is singular. Therefore, coefficients c_q cannot be reconstructed by solving (10). The solution suggested is the concatenation of $\mathbf{AH}^2$ to $\mathbf{AH}^1$. However, examining $\mathbf{AH}^2$ for, say $kr = 1.9$ reveals that $\mathbf{AH}^2$ is

$$\mathbf{AH}^2 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.32 & 0 & -0.32 \\ 0 & 0 & 0.64 & 0 \\ 0 & -0.32 & 0 & 0.32 \end{bmatrix} \tag{16}$$

It is evident that the second column of $\mathbf{AH}^2$ is equal to the forth column times minus one, so that after concatenating the matrix is still rank deficient and hence the function cannot be reconstructed. It is worth noting that adding more spheres would not increase the maximum order which can be sampled without aliasing.

The reason for the rank deficiency is now explained: as proved in the appendix (and can be seen clearly in the last example) for equiangular sampling the columns in $\mathbf{A}$ corresponding to $q = (n,m) = (n, N+1)$ equals to the $(n, -N-1)$ column times $(-1)^{N+1}$. In addition, matrix $\mathbf{H}^1$ is diagonal with elements $j_n(kr_1)$ independent of m, so the columns corresponding to $(n, N+1)$ and to $(n, -N-1)$ are both multiplied with the same value. As for $\mathbf{H}^2$, the relation between the two columns is similar, multiplied with $j_n(kr_2)$,

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so that after concatenating these columns remain linearly dependent.

In conclusion, the dual-sphere configuration is not efficient in terms of the number of samples. Furthermore, the above development shows that employing more spheres with different radii would not increase the maximum order that can be sampled without aliasing.

3.3. Efficient Dual-Sphere Configuration

A more efficient way to maximum the order in terms of the number of microphones for the dual-sphere sampling method is now suggested. As mentioned, a consequence of the GSE framework is that the maximum spherical order which can be sampled does not increase when employing more systems, if these system show similar behavior along m for a given value of n. Therefore, in order to achieve a higher spatial resolution one sphere is rotated relative to the other so that the system show different behavior along m due to the rotation operation. This way the two spheres can be represented as two linear systems, one with a radius r_1, while the other has a radius r_2 and is rotated relative to the first. The coefficients $h_{qq'}$ for the rotated system are $D_{qq'}(\alpha, \beta, \gamma)$ and are the Wigner-D functions defined in [4], where (α, β, γ) are the Euler angles at which the is sphere rotated.

Analysis of the Wigner-D functions reveals that in order to increase the maximum order which can be computed the angle β must not be zero or π. Simulation results, presented in figure 3, showed that for a rotation about the x-axis, i.e. $\alpha = \gamma = 0$, the best condition number is achieved for $\pi/2$ and $3\pi/2$, while for $0, \pi$ the matrix $[\mathbf{AH}]$ is rank deficient so $\mathbf{p}$ could not be computed.

Another way to increase the maximum order which can be sampled is mentioned. As showed in the appendix, the $(n, N+1)$ and the $(n, -N-1)$ columns are dependent due to the symmetry property of the equiangular sampling scheme. This is true as well for the Gaussian sampling scheme, but not for the nearly-uniform sampling scheme. Simulations showed that it is possible to increase the maximal order that can be sampled without error when using the dual-sphere configuration with a nearly-uniform sampling scheme.

4. CONCLUSIONS

In this paper we developed the theory of the reconstruction of an aliased function using a series of linear systems within a spherical coordinate system by applying the general sampling expansion (GSE) to the spherical harmonics expansion. Studies of both linear systems and the aliasing effect in the spherical harmonics domain have been presented. The GSE is applied to the dual sphere sampling method in order to overcome ill-conditioning effects at specific frequencies due to the nulls of the spherical Bessel function and, simultaneously, in order to increase the array order. The increased order is achieved only when one of the two spheres is rotated relative to the other.

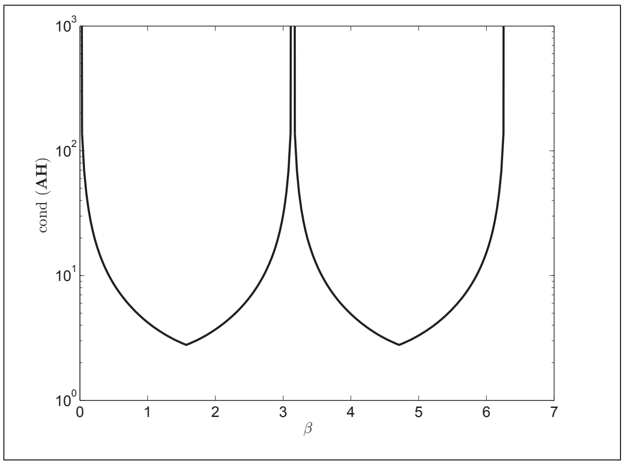

Figure 3. Condition number as a function of rotation about the x axis for the dual sphere, sampled with equiangular scheme. Minimum condition number is achieved at $\beta = \pi/2, 3\pi/2$.

Appendix

We present a short proof which shows that when sampling a function using equiangular sampling with an array of order N, the reconstructed coefficients $\hat{p}_n^{N+1}$ are equal to $\hat{p}_n^{-N-1}$ and hence the aliasing matrix $\mathbf{A}$ has linearly dependent lines.

The spherical coefficients p_n^m are notated in this section with a double index (n, m) where $q = n^2+n+m$. The spherical harmonic $Y_n^m(\theta, \phi)$ is defined by:

$$Y_n^m(\theta, \phi) \triangleq \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi}$$

where n is the order of the spherical harmonics, $P_n^m(\cdot)$ is the associated Legendre function, and $i = \sqrt{-1}$.

The aliasing matrix defined in section 2.2 for the equiangular sampling is [4]

$$\begin{aligned}
&\mathbf{A}_{nm,n'm'} \\
&= \mu_{n,m}\eta_{n',m'} \sum_{k=0}^{2N+1}\sum_{j=0}^{2N+1} \alpha_k Y_{n'}^{m'}(\theta_k, \phi_j) Y_n^{m*}(\theta_k, \phi_j) \\
&= \mu_{n,m}\mu_{n',m'} \sum_{k=0}^{2N+1} \alpha_k P_n^m(\cos\theta_k) P_{n'}^{m'}(\cos\theta_k) \\
&\quad \times \sum_{j=0}^{2N+1} e^{i(m'-m)\phi_j}
\end{aligned} \quad \text{(A1)}$$

where $\theta_k = \pi k/2(N+1)$ and $\phi_j = 2\pi j/2(N+1)$ and $\mu_{n,m}$ is

$$\sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}$$

The spherical harmonics satisfies $Y_n^{-m} = (-1)^m Y_n^{m*}$, so that for negative m' (A1) becomes

$$\begin{aligned} &\mathbf{A}_{nm,n'(-m')} \\ &= (-1)^{m'} \sum_{k=0}^{2N} \alpha_k P_n^m(\cos\theta_k) P_{n'}^{m'}(\cos\theta_k) \\ &\quad \times \sum_{j=0}^{2N} e^{i(-m'-m)\phi_j} \end{aligned} \tag{A2}$$

The function $e^{i(m'-m)\phi_j}$ is periodic over m' with a period of $2N + 2$. Hence $e^{i(-(N+1)-m)\phi_j} = e^{i(N+1-m)\phi_j}$, which, using (A1) and (A2) yields

$$\mathbf{A}_{nm,n'(-N-1)} = (-1)^{N+1} \mathbf{A}_{nm,n'(N+1)} \tag{A3}$$

This shows that $\mathbf{A}$ has linearly dependent columns when employing a sampling function of order higher than N, causing erroneous reconstruction of $\hat{p}_n^{N+1}$ and $\hat{p}_n^{-N-1}$.

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